Introduction/Plan

Airy functions were named after George Biddel Airy (1801-1892), who excelled in the fields of Applied mathematics and Astrophysics over the course of his life. He was president of the Royal Astronomical Society for four complete terms and has a lunar crater named in his honour, as well as some other mathematical effects, such as Airy discs and Airy points- but for the purposes of this project the Airy Equations and their solutions will be the point of focus.

Airy functions are the solutions to the Airy Equation

For which there are two linearly independent solutions. These are commonly denoted as Ai(x), and Bi(x) but sometimes can be found to be denoted with trigonometric prefixes ie … due to the solutions relationships with some trigonometric functions, I will use the former throughout this project.

One of the simplest way of representing these solutions is in integral form, first used by Jeffreys

where the relationship to Sine and Cosine becomes apparent.

The Airy Equation (1) is most famous for being the simplest second order linear ordinary differential equation which has a turning point. Due to this particular characteristic, it is used across many disciplines in modelling particle behaviour in physics, to optics and beyond.

The purpose of this project is to explore some techniques of Applied Mathematics so find methods of numerically approximating these functions. Chapter 2 will discuss first the relationship of these solutions to Bessel Functions of order 1/3. This is an important and convenient connection, as the numerical expansions, and more importantly for this Chapter is the asymptotic expansions of Bessel Functions. Asymptotic Expansions are powerful tools, as they do not compound iterative errors in solutions that you would find in a finite order approximation through a Taylor Expansion, as you will see in a direct comparison of analytical, asymptotic and a vanilla Taylor expansion.

Asymptotic methods are powerful and unreasonably accurate for large values, but we will see that the critical behaviours of the solutions are not preserved at the critical point of the analytical solution. Hence in Chapter 3, I will discuss a different technique which will hopefully give a stronger numerical approximation about the origin using a WKB (Wentzel–Kramers–Brillouin) method, which we will compare to the other numerical, and analytical solutions discussed.

Chapter 4 will try to combine the numerical solutions of these methods to try and capture the behaviour of the solutions about the turning point while maintaining the high precision of the first technique.

Chapter 5 will be a conclusion, discussing the accuracy of the final solution and where any improvements could be made on it’s accuracy.

Where σ is a large but finite number. We now substitute this integral into (1).

As the function oscillates between and . Thus the mean value of the function can be set to zero. Now in this limit, is a solution to (1). Now the integral is known as the Airy integral, ie

Deriving power series

Differentiate twice:

(

(

Now change the index of the sum to start with k=1:

)

Since , must be zero, and furthermore . Therefore can be expressed as

The values of the coefficients can now be calculated in steps of three, starting with . Hence

The two linearly independent solutions are then generated from the arbitrary constants, and . This is demonstrated by writing out then full power series and grouping the terms by and .

, …

and

Hence

Study of Caustics led airy to the functions

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